

# An Extremal Problem On Potentially $K_{r+1} - H$ -graphic Sequences <sup>\*</sup>

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## Abstract

Let  $K_k$ ,  $C_k$ ,  $T_k$ , and  $P_k$  denote a complete graph on  $k$  vertices, a cycle on  $k$  vertices, a tree on  $k + 1$  vertices, and a path on  $k + 1$  vertices, respectively. Let  $K_m - H$  be the graph obtained from  $K_m$  by removing the edges set  $E(H)$  of the graph  $H$  ( $H$  is a subgraph of  $K_m$ ). A sequence  $S$  is potentially  $K_m - H$ -graphical if it has a realization containing a  $K_m - H$  as a subgraph. Let  $\sigma(K_m - H, n)$  denote the smallest degree sum such that every  $n$ -term graphical sequence  $S$  with  $\sigma(S) \geq \sigma(K_m - H, n)$  is potentially  $K_m - H$ -graphical. In this paper, we determine the values of  $\sigma(K_{r+1} - H, n)$  for  $n \geq 4r + 10, r \geq 3, r + 1 \geq k \geq 4$  where  $H$  is a graph on  $k$  vertices which contains a tree on 4 vertices but not contains a cycle on 3 vertices. We also determine the values of  $\sigma(K_{r+1} - P_2, n)$  for  $n \geq 4r + 8, r \geq 3$ .

**Key words:** graph; degree sequence; potentially  $K_{r+1} - H$ -graphic sequence

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## 1 Introduction

The set of all non-increasing nonnegative integers sequence  $\pi = (d(v_1), d(v_2), \dots, d(v_n))$  is denoted by  $NS_n$ . A sequence  $\pi \in NS_n$  is said to be graphic if it is the degree sequence of a simple graph  $G$  on  $n$  vertices, and

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such a graph  $G$  is called a realization of  $\pi$ . The set of all graphic sequences in  $NS_n$  is denoted by  $GS_n$ . A graphical sequence  $\pi$  is potentially  $H$ -graphical if there is a realization of  $\pi$  containing  $H$  as a subgraph, while  $\pi$  is forcibly  $H$ -graphical if every realization of  $\pi$  contains  $H$  as a subgraph. If  $\pi$  has a realization in which the  $r+1$  vertices of largest degree induce a clique, then  $\pi$  is said to be potentially  $A_{r+1}$ -graphic. Let  $\sigma(\pi) = d(v_1) + d(v_2) + \dots + d(v_n)$ , and  $[x]$  denote the largest integer less than or equal to  $x$ . We denote  $G + H$  as the graph with  $V(G + H) = V(G) \cup V(H)$  and  $E(G + H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$ . Let  $K_k$ ,  $C_k$ ,  $T_k$ , and  $P_k$  denote a complete graph on  $k$  vertices, a cycle on  $k$  vertices, a tree on  $k+1$  vertices, and a path on  $k+1$  vertices, respectively. Let  $K_m - H$  be the graph obtained from  $K_m$  by removing the edges set  $E(H)$  of the graph  $H$  ( $H$  is a subgraph of  $K_m$ ).

Given a graph  $H$ , what is the maximum number of edges of a graph with  $n$  vertices not containing  $H$  as a subgraph? This number is denoted  $ex(n, H)$ , and is known as the Turán number. This problem was proposed for  $H = C_4$  by Erdős [2] in 1938 and generalized by Turán [15]. In terms of graphic sequences, the number  $2ex(n, H) + 2$  is the minimum even integer  $l$  such that every  $n$ -term graphical sequence  $\pi$  with  $\sigma(\pi) \geq l$  is forcibly  $H$ -graphical. Here we consider the following variant: determine the minimum even integer  $l$  such that every  $n$ -term graphical sequence  $\pi$  with  $\sigma(\pi) \geq l$  is potentially  $H$ -graphical. We denote this minimum  $l$  by  $\sigma(H, n)$ . Erdős, Jacobson and Lehel [4] showed that  $\sigma(K_k, n) \geq (k-2)(2n-k+1) + 2$  and conjectured that equality holds. They proved that if  $\pi$  does not contain zero terms, this conjecture is true for  $k = 3$ ,  $n \geq 6$ . The conjecture is confirmed in [5], [10], [11], [12] and [13].

Gould, Jacobson and Lehel [5] also proved that  $\sigma(pK_2, n) = (p-1)(2n-2) + 2$  for  $p \geq 2$ ;  $\sigma(C_4, n) = 2\lceil \frac{3n-1}{2} \rceil$  for  $n \geq 4$ . Luo [14] characterized the potentially  $C_k$  graphic sequence for  $k = 3, 4, 5$ . Lai [7] determined  $\sigma(K_4 - e, n)$  for  $n \geq 4$ . Lai [8, 9] determined  $\sigma(K_5 - C_4, n)$ ,  $\sigma(K_5 - P_3, n)$  and  $\sigma(K_5 - P_4, n)$ , for  $n \geq 5$ . Yin, Li and Mao [17] determined  $\sigma(K_{r+1} - e, n)$  for  $r \geq 3$ ,  $r+1 \leq n \leq 2r$  and  $\sigma(K_5 - e, n)$  for  $n \geq 5$ . Yin and Li [16] gave a good method (Yin-Li method) of determining the values  $\sigma(K_{r+1} - e, n)$  for  $r \geq 2$  and  $n \geq 3r^2 - r - 1$ . After reading [16], using Yin-Li method Yin [18] determined the values  $\sigma(K_{r+1} - K_3, n)$  for  $r \geq 3$ ,  $n \geq 3r + 5$ . Determining  $\sigma(K_{r+1} - H, n)$ , where  $H$  is a tree on 4 vertices is more useful than a cycle on 4 vertices (for example,  $C_4 \not\subset C_i$ , but  $P_3 \subset C_i$  for  $i \geq 5$ ). So, after reading [16] and [18], using Yin-Li method we prove the following three theorems.

**Theorem 1.1.** If  $r \geq 3$  and  $n \geq 4r + 8$ , then  $\sigma(K_{r+1} - P_2, n) = (r-1)(2n-r) - 2(n-r) + 2$ .

**Theorem 1.2.** If  $r \geq 3$  and  $n \geq 4r + 10$ , then  $\sigma(K_{r+1} - T_3, n) = (r-1)(2n-r) - 2(n-r)$ .

**Theorem 1.3.** If  $r \geq 3, r+1 \geq k \geq 4$  and  $n \geq 4r+10$ , then  $\sigma(K_{r+1} - H, n) = (r-1)(2n-r) - 2(n-r)$ , where  $H$  is a graph on  $k$  vertices which contains a tree on 4 vertices but not contains a cycle on 3 vertices.

There are a number of graphs on  $k$  vertices which containing a tree on 4 vertices but not containing a cycle on 3 vertices (for example, the cycle on  $k$  vertices, the tree on  $k$  vertices, and the complete 2-partite graph on  $k$  vertices, etc ).

## 2 Preparations

In order to prove our main result, we need the following notations and results.

Let  $\pi = (d_1, \dots, d_n) \in NS_n, 1 \leq k \leq n$ . Let

$$\pi''_k = \begin{cases} (d_1 - 1, \dots, d_{k-1} - 1, d_{k+1} - 1, \dots, d_{d_k+1} - 1, d_{d_k+2}, \dots, d_n), \\ \text{if } d_k \geq k, \\ (d_1 - 1, \dots, d_{d_k} - 1, d_{d_k+1}, \dots, d_{k-1}, d_{k+1}, \dots, d_n), \\ \text{if } d_k < k. \end{cases}$$

Denote  $\pi'_k = (d'_1, d'_2, \dots, d'_{n-1})$ , where  $d'_1 \geq d'_2 \geq \dots \geq d'_{n-1}$  is a rearrangement of the  $n-1$  terms of  $\pi''_k$ . Then  $\pi'_k$  is called the residual sequence obtained by laying off  $d_k$  from  $\pi$ .

**Theorem 2.1[16]** Let  $n \geq r+1$  and  $\pi = (d_1, d_2, \dots, d_n) \in GS_n$  with  $d_{r+1} \geq r$ . If  $d_i \geq 2r-i$  for  $i = 1, 2, \dots, r-1$ , then  $\pi$  is potentially  $A_{r+1}$ -graphic.

**Theorem 2.2[16]** Let  $n \geq 2r+2$  and  $\pi = (d_1, d_2, \dots, d_n) \in GS_n$  with  $d_{r+1} \geq r$ . If  $d_{2r+2} \geq r-1$ , then  $\pi$  is potentially  $A_{r+1}$ -graphic.

**Theorem 2.3[16]** Let  $n \geq r+1$  and  $\pi = (d_1, d_2, \dots, d_n) \in GS_n$  with  $d_{r+1} \geq r-1$ . If  $d_i \geq 2r-i$  for  $i = 1, 2, \dots, r-1$ , then  $\pi$  is potentially  $K_{r+1} - e$ -graphic.

**Theorem 2.4[16]** Let  $n \geq 2r+2$  and  $\pi = (d_1, d_2, \dots, d_n) \in GS_n$  with  $d_{r-1} \geq r$ . If  $d_{2r+2} \geq r-1$ , then  $\pi$  is potentially  $K_{r+1} - e$ -graphic.

**Theorem 2.5[6]** Let  $\pi = (d_1, \dots, d_n) \in NS_n$  and  $1 \leq k \leq n$ . Then  $\pi \in GS_n$  if and only if  $\pi'_k \in GS_{n-1}$ .

**Theorem 2.6[3]** Let  $\pi = (d_1, \dots, d_n) \in NS_n$  with even  $\sigma(\pi)$ . Then  $\pi \in GS_n$  if and only if for any  $t, 1 \leq t \leq n-1$ ,

$$\sum_{i=1}^t d_i \leq t(t-1) + \sum_{j=t+1}^n \min\{t, d_j\}.$$

**Theorem 2.7[5]** If  $\pi = (d_1, d_2, \dots, d_n)$  is a graphic sequence with a realization  $G$  containing  $H$  as a subgraph, then there exists a realization

$G'$  of  $\pi$  containing  $H$  as a subgraph so that the vertices of  $H$  have the largest degrees of  $\pi$ .

**Lemma 2.1** [18] If  $\pi = (d_1, d_2, \dots, d_n) \in NS_n$  is potentially  $K_{r+1} - e$ -graphic, then there is a realization  $G$  of  $\pi$  containing  $K_{r+1} - e$  with the  $r+1$  vertices  $v_1, \dots, v_{r+1}$  such that  $d_G(v_i) = d_i$  for  $i = 1, 2, \dots, r+1$  and  $e = v_r v_{r+1}$ .

**Lemma 2.2** [18] If  $r \geq 3$  and  $n \geq r+1$ , then  $\sigma(K_{r+1} - K_3, n) \geq (r-1)(2n-r) - 2(n-r) + 2$ .

### 3 Proof of Main results.

**Lamma 3.1** Let  $n \geq r+1$  and  $\pi = (d_1, d_2, \dots, d_n) \in GS_n$  with  $d_r \geq r-1$  and  $d_{r+1} \geq r-2$ . If  $d_i \geq 2r-i$  for  $i = 1, 2, \dots, r-2$ , then  $\pi$  is potentially  $K_{r+1} - P_2$ -graphic.

**Proof.** We consider the following two cases.

Case 1:  $d_{r+1} \geq r-1$ .

Subcase 1.1:  $d_{r-1} \geq r+1$ . Then  $\pi$  is potentially  $K_{r+1} - e$ -graphic by Theorem 2.3. Hence,  $\pi$  is potentially  $K_{r+1} - P_2$ -graphic.

Subcase 1.2:  $d_{r-1} = r-1$ . Then  $d_{r-1} = d_r = d_{r+1} = r-1$ .

If  $d_{r+2} = r-1$ , then the residual sequence  $\pi'_{r+1} = (d'_1, \dots, d'_{n-1})$  obtained by laying off  $d_{r+1} = r-1$  from  $\pi$  satisfies: (1)  $d'_i = d_i - 1$  for  $i = 1, 2, \dots, r-2$ , (2)  $d'_1 \geq 2(r-1) - 1, \dots, d'_{(r-1)-1} = d'_{r-2} \geq 2(r-1) - (r-2), d'_{r-1} = d_r, d'_{(r-1)+1} = d'_r = d_{r+2} = r-1$ . By Theorem 2.1,  $\pi'_{r+1}$  is potentially  $A_{(r-1)+1}$ -graphic. Therefore,  $\pi$  is potentially  $K_{r+1} - P_2$ -graphic by  $\{d_1 - 1, \dots, d_{r-2} - 1, d_r, d_{r+2}\} = \{d'_1, \dots, d'_r\}$  and Theorem 2.7.

If  $d_{r+2} \leq r-2$ , then the residual sequence  $\pi'_{r+1} = (d'_1, \dots, d'_{n-1})$  obtained by laying off  $d_{r+1} = r-1$  from  $\pi$  satisfies: (1)  $d'_i = d_i - 1$  for  $i = 1, 2, \dots, r-2$ , (2)  $d'_1 \geq 2(r-1) - 1, \dots, d'_{(r-1)-1} = d'_{r-2} \geq 2(r-1) - (r-2), d'_{r-1} = d_r, d'_{(r-1)+1} = d'_r = d_{r-1} - 1 = r-2$ . By Theorem 2.3,  $\pi'_{r+1}$  is potentially  $K_{(r-1)+1} - e$ -graphic. Therefore,  $\pi$  is potentially  $K_{r+1} - P_2$ -graphic by  $\{d_1 - 1, \dots, d_{r-2} - 1, d_r, d_{r-1} - 1\} = \{d'_1, \dots, d'_r\}$  and Lemma 2.1.

Subcase 1.3:  $d_{r-1} = r$ . Then  $d_{r+1} = r$  or  $r-1$ .

If  $d_{r+1} = r$ , then  $d_{r-1} = d_r = d_{r+1} = r$ . The residual sequence  $\pi'_{r+1}$  satisfies: (1)  $d'_i = d_i - 1$  for  $i = 1, 2, \dots, r-2$ , (2)  $d'_1 \geq d_1 - 1 \geq 2(r-1) - 1, \dots, d'_{(r-1)-1} = d'_{r-2} \geq d_{r-2} - 1 \geq 2(r-1) - (r-2)$  and  $d'_{(r-1)+1} = d'_r \geq d_r - 1 = r-1$ . By Theorem 2.1,  $\pi'_{r+1}$  is potentially  $A_{(r-1)+1}$ -graphic. Thus,  $\pi$  is potentially  $K_{r+1} - P_2$ -graphic by  $\{d_1 - 1, \dots, d_{r-2} - 1\} \subseteq \{d'_1, \dots, d'_r\}$  and Theorem 2.7.

If  $d_{r+1} = r-1$ , then  $d_r = r-1$  or  $r$ .

If  $d_r = r - 1$ , then  $\pi'_{r+1}$  satisfies: (1)  $d'_i = d_i - 1$  for  $i = 1, 2, \dots, r - 1$ , (2)  $d'_1 \geq d_1 - 1 \geq 2(r - 1) - 1, \dots, d'_{(r-1)-1} = d'_{r-2} = d_{r-2} - 1 \geq 2(r - 1) - (r - 2)$  and  $d'_{(r-1)+1} = d'_r = d_r = r - 1$ . According to Theorem 2.1,  $\pi'_{r+1}$  is potentially  $A_{(r-1)+1}$ -graphic. Therefore,  $\pi$  is potentially  $K_{r+1} - P_2$ -graphic by  $\{d_1 - 1, \dots, d_{r-1} - 1, d_r\} = \{d'_1, \dots, d'_r\}$  and Theorem 2.7.

If  $d_r = r$ , then  $\pi'_{r+1}$  satisfies: (1)  $d'_i = d_i - 1$  for  $i = 1, 2, \dots, r - 2$ , (2)  $d'_1 \geq d_1 - 1 \geq 2(r - 1) - 1, \dots, d'_{(r-1)-1} = d'_{r-2} = d_{r-2} - 1 \geq 2(r - 1) - (r - 2)$  and  $d'_{(r-1)+1} = d'_r = d_{r-1} - 1 = r - 1$ . By Theorem 2.1,  $\pi'_{r+1}$  is potentially  $A_{(r-1)+1}$ -graphic. Therefore,  $\pi$  is potentially  $K_{r+1} - P_2$ -graphic by  $\{d_1 - 1, \dots, d_{r-2} - 1, d_r, d_{r-1} - 1\} = \{d'_1, \dots, d'_r\}$  and Theorem 2.7.

Case 2:  $d_{r+1} \leq r - 2$ , that is,  $d_{r+1} = r - 2$ .

If  $d_{r-1} < d_{r-2}$ , then  $\pi'_{r+1}$  satisfies: (1)  $d'_i = d_i - 1$  for  $i = 1, 2, \dots, r - 2$ , (2)  $d'_1 \geq d_1 - 1 \geq 2(r - 1) - 1, \dots, d'_{(r-1)-1} = d'_{r-2} = d_{r-2} - 1 \geq 2(r - 1) - [(r - 1) - 1]$  and  $d'_{(r-1)+1} = d'_r = d_r \geq r - 1$ . By Theorem 2.1,  $\pi'_{r+1}$  is potentially  $A_{(r-1)+1}$ -graphic. Therefore,  $\pi$  is potentially  $K_{r+1} - P_2$ -graphic by  $\{d_1 - 1, \dots, d_{r-2} - 1, d_{r-1}, d_r\} = \{d'_1, \dots, d'_r\}$  and Theorem 2.7.

If  $d_{r-1} = d_{r-2} \geq r + 2$ , then  $\pi'_{r+1}$  satisfies:  $d'_1 \geq d_1 - 1 \geq 2(r - 1) - 1, \dots, d'_{(r-1)-1} = d'_{r-2} \geq d_{r-2} - 1 \geq 2(r - 1) - [(r - 1) - 1]$  and  $d'_{(r-1)+1} = d'_r \geq r - 1$ . By Theorem 2.1,  $\pi'_{r+1}$  is potentially  $A_{(r-1)+1}$ -graphic. Therefore,  $\pi$  is potentially  $K_{r+1} - P_2$ -graphic by  $\{d_{r-1}, d_r, d_1 - 1, \dots, d_{r-2} - 1\} = \{d'_1, \dots, d'_r\}$  and Theorem 2.7.

**Lemma 3.2.** Let  $n \geq 2r + 2$  and  $\pi = (d_1, d_2, \dots, d_n) \in GS_n$  with  $d_{r-2} \geq r$ . If  $d_{2r+2} \geq r - 1$ , then  $\pi$  is potentially  $K_{r+1} - P_2$ -graphic.

**Proof.** We consider the following two cases.

Case 1: If  $d_{r-1} \geq r$ . Then  $\pi$  is potentially  $K_{r+1} - e$ -graphic by Theorem 2.4. Hence,  $\pi$  is potentially  $K_{r+1} - P_2$ -graphic.

Case 2:  $d_{r-1} \leq r - 1$ , that is,  $d_{r-1} = r - 1$ , then  $d_{r-1} = d_r = d_{r+1} = \dots = d_{2r+2} = r - 1$  and  $\pi'_{r+1}$  satisfies: (1)  $d'_i = d_i - 1$  for  $i = 1, 2, \dots, r - 2$ , (2)  $d'_{(r-1)+1} = d'_r \geq r - 1$  and  $d'_{2(r-1)+2} = d'_{2r} \geq (r - 1) - 1$ . By Theorem 2.2,  $\pi'_{r+1}$  is potentially  $A_r$ -graphic. Therefore,  $\pi$  is potentially  $K_{r+1} - P_2$ -graphic by  $\{d_1 - 1, \dots, d_{r-2} - 1, d_r, d_{r+2}\} = \{d'_1, \dots, d'_r\}$  and Theorem 2.7.

**Lemma 3.3.** If  $r \geq 3$  and  $n \geq r + 1$ , then  $\sigma(K_{r+1} - P_2, n) \geq (r - 1)(2n - r) - 2(n - r) + 2$ .

**Proof.** By Lemma 2.2, for  $r \geq 3$  and  $n \geq r + 1$ ,  $\sigma(K_{r+1} - K_3, n) \geq (r - 1)(2n - r) - 2(n - r) + 2$ . Obviously, for  $r \geq 3$  and  $n \geq r + 1$ ,  $\sigma(K_{r+1} - P_2, n) \geq \sigma(K_{r+1} - K_3, n) \geq (r - 1)(2n - r) - 2(n - r) + 2$ .

**Lemma 3.4.** If  $r \geq 3, r + 1 \geq k \geq 4$  and  $n \geq r + 1$ , then  $\sigma(K_{r+1} - H, n) \geq (r - 1)(2n - r) - 2(n - r)$ , for  $H$  be a graph on  $k$  vertices which containing a tree on 4 vertices but not containing a cycle on 3 vertices.

**Proof.** Let

$$G = K_{r-2} + \overline{K_{n-r+2}}$$

Then  $G$  is a unique realization of  $((n-1)^{r-2}, (r-2)^{n-r+2})$  and  $G$  clearly does not contain  $K_{r+1} - H$ , where the symbol  $x^y$  means  $x$  repeats  $y$  times in the sequence. Thus

$$\sigma(K_{r+1} - H, n) \geq (r-2)(n-1) + (r-2)(n-r+2) + 2 = (r-1)(2n-r) - 2(n-r).$$

**The Proof of Theorem 1.1** According to Lemma 3.3, it is enough to verify that for  $r \geq 3$  and  $n \geq 4r + 8$ ,

$$\sigma(K_{r+1} - P_2, n) \leq (r-1)(2n-r) - 2(n-r) + 2.$$

We now prove that if  $n \geq 4r + 8$  and  $\pi = (d_1, d_2, \dots, d_n) \in GS_n$  with

$$\sigma(\pi) \geq (r-1)(2n-r) - 2(n-r) + 2,$$

then  $\pi$  is potentially  $K_{r+1} - P_2$ -graphic.

If  $d_{r-2} \leq r-1$ , then

$$\begin{aligned} \sigma(\pi) &\leq (r-3)(n-1) + (r-1)(n-r+3) \\ &= (r-1)(n-1) - 2(n-1) + (r-1)(n-r+3) \\ &= (r-1)(2n-r) - 2(n-r) \\ &< (r-1)(2n-r) - 2(n-r) + 2, \end{aligned}$$

which is a contradiction. Thus  $d_{r-2} \geq r$ .

If  $d_r \leq r-2$ , then

$$\begin{aligned} \sigma(\pi) &= \sum_{i=1}^{r-1} d_i + \sum_{i=r}^n d_i \\ &\leq (r-1)(r-2) + \sum_{i=r}^n \min\{r-1, d_i\} + \sum_{i=r}^n d_i \\ &= (r-1)(r-2) + 2 \sum_{i=r}^n d_i \\ &\leq (r-1)(r-2) + 2(n-r+1)(r-2) \\ &= (r-1)(2n-r) - 2(n-r) - 2 \\ &< (r-1)(2n-r) - 2(n-r) + 2, \end{aligned}$$

which is a contradiction. Hence  $d_r \geq r-1$ .

If  $d_{r+1} \leq r-3$ , then

$$\begin{aligned} \sigma(\pi) &= \sum_{i=1}^{r-1} d_i + d_r + \sum_{i=r+1}^n d_i \\ &\leq (r-1)(r-2) + \sum_{i=r}^n \min\{r-1, d_i\} + d_r + \sum_{i=r+1}^n d_i \\ &= (r-1)(r-2) + \min\{r-1, d_r\} + d_r + 2 \sum_{i=r+1}^n d_i \\ &\leq (r-1)(r-2) + 2d_r + 2 \sum_{i=r+1}^n d_i \\ &\leq (r-1)(r-2) + 2(n-1) + 2(n-r)(r-3) \\ &= (r-1)(2n-r) - 2(n-r) \\ &< (r-1)(2n-r) - 2(n-r) + 2, \end{aligned}$$

which is a contradiction. Thus  $d_{r+1} \geq r-2$ .

If  $d_i \geq 2r - i$  for  $i = 1, 2, \dots, r - 2$  or  $d_{2r+2} \geq r - 1$ , then  $\pi$  is potentially  $K_{r+1} - P_2$ -graphic by Lemma 3.1 or Lemma 3.2. If  $d_{2r+2} \leq r - 2$  and there exists an integer  $i, 1 \leq i \leq r - 2$  such that  $d_i \leq 2r - i - 1$ , then

$$\begin{aligned}\sigma(\pi) &\leq (i - 1)(n - 1) + (2r + 1 - i + 1)(2r - i - 1) \\ &\quad + (r - 2)(n + 1 - 2r - 2) \\ &= i^2 + i(n - 4r - 2) - (n - 1) + (2r - 1)(2r + 2) \\ &\quad + (r - 2)(n - 2r - 1).\end{aligned}$$

Since  $n \geq 4r + 8$ , it is easy to see that  $i^2 + i(n - 4r - 2)$ , consider as a function of  $i$ , attains its maximum value when  $i = r - 2$ . Therefore,

$$\begin{aligned}\sigma(\pi) &\leq (r - 2)^2 + (n - 4r - 2)(r - 2) - (n - 1) \\ &\quad + (2r - 1)(2r + 2) + (r - 2)(n - 2r - 1) \\ &= (r - 1)(2n - r) - 2(n - r) + 2 - n + 4r + 7 \\ &< \sigma(\pi),\end{aligned}$$

which is a contradiction.

Thus,  $\sigma(K_{r+1} - P_2, n) \leq (r - 1)(2n - r) - 2(n - r) + 2$  for  $n \geq 4r + 8$ .

**The Proof of Theorem 1.2** According to Lemma 3.4, it is enough to verify that for  $r \geq 3$  and  $n \geq 4r + 10$ ,

$$\sigma(K_{r+1} - T_3, n) \leq (r - 1)(2n - r) - 2(n - r).$$

We now prove that if  $n \geq 4r + 10$  and  $\pi = (d_1, d_2, \dots, d_n) \in GS_n$  with

$$\sigma(\pi) \geq (r - 1)(2n - r) - 2(n - r),$$

then  $\pi$  is potentially  $K_{r+1} - T_3$ -graphic.

If  $d_{r-2} \leq r - 1$ , we consider the following cases.

(1) Suppose  $d_{r-2} = r - 1$  and  $\sigma(\pi) = (r - 3)(n - 1) + (r - 1)(n - r + 3)$ , then  $\pi = ((n - 1)^{r-3}, (r - 1)^{n-r+3})$ . Obviously  $\pi$  is potentially  $K_{r+1} - T_3$  graphic.

(2) Suppose  $d_{r-2} = r - 1$  and  $\sigma(\pi) < (r - 3)(n - 1) + (r - 1)(n - r + 3)$ , then

$$\begin{aligned}\sigma(\pi) &< (r - 3)(n - 1) + (r - 1)(n - r + 3) \\ &= (r - 1)(n - 1) - 2(n - 1) + (r - 1)(n - r + 3) \\ &= (r - 1)(2n - r) - 2(n - r),\end{aligned}$$

which is a contradiction.

(3) Suppose  $d_{r-2} < r - 1$ , then

$$\begin{aligned}\sigma(\pi) &< (r - 3)(n - 1) + (r - 1)(n - r + 3) \\ &= (r - 1)(n - 1) - 2(n - 1) + (r - 1)(n - r + 3) \\ &= (r - 1)(2n - r) - 2(n - r),\end{aligned}$$

which is a contradiction.

Thus,  $d_{r-2} \geq r$  or  $\pi$  is potentially  $K_{r+1} - T_3$  graphic.

If  $d_r \leq r - 2$ , then

$$\begin{aligned}
\sigma(\pi) &= \sum_{i=1}^{r-1} d_i + \sum_{i=r}^n d_i \\
&\leq (r-1)(r-2) + \sum_{i=r}^n \min\{r-1, d_i\} + \sum_{i=r}^n d_i \\
&= (r-1)(r-2) + 2 \sum_{i=r}^n d_i \\
&\leq (r-1)(r-2) + 2(n-r+1)(r-2) \\
&= (r-1)(2n-r) - 2(n-r) - 2 \\
&< (r-1)(2n-r) - 2(n-r),
\end{aligned}$$

which is a contradiction. Hence  $d_r \geq r - 1$ .

If  $d_{r+1} \leq r - 3$ , we consider the following cases.

(1) Suppose  $d_r = n - 1$ , then  $d_1 \geq d_2 \geq \dots \geq d_{r-1} \geq d_r = n - 1$ , therefore  $d_1 = d_2 = \dots = d_r = n - 1$ . Therefore  $d_{r+1} \geq r$ , which is a contradiction.

(2) Suppose  $d_r \leq n - 2$ , then

$$\begin{aligned}
\sigma(\pi) &= \sum_{i=1}^{r-1} d_i + d_r + \sum_{i=r+1}^n d_i \\
&\leq (r-1)(r-2) + \sum_{i=r}^n \min\{r-1, d_i\} + d_r + \sum_{i=r+1}^n d_i \\
&= (r-1)(r-2) + \min\{r-1, d_r\} + d_r + 2 \sum_{i=r+1}^n d_i \\
&\leq (r-1)(r-2) + 2d_r + 2 \sum_{i=r+1}^n d_i \\
&\leq (r-1)(r-2) + 2(n-2) + 2(n-r)(r-3) \\
&= (r-1)(2n-r) - 2(n-r) - 2 \\
&< (r-1)(2n-r) - 2(n-r),
\end{aligned}$$

which is a contradiction.

Thus  $d_{r+1} \geq r - 2$ .

If  $d_i \geq 2r - i$  for  $i = 1, 2, \dots, r - 2$  or  $d_{2r+2} \geq r - 1$ , then  $\pi$  is potentially  $K_{r+1} - T_3$  graphic ( $\pi = ((n-1)^{r-3}, (r-1)^{n-r+3})$ ) or  $\pi$  is potentially  $K_{r+1} - P_2$ -graphic by Lemma 3.1 or Lemma 3.2. Therefore,  $\pi$  is potentially  $K_{r+1} - T_3$ -graphic. If  $d_{2r+2} \leq r - 2$  and there exists an integer  $i$ ,  $1 \leq i \leq r - 2$  such that  $d_i \leq 2r - i - 1$ , then

$$\begin{aligned}
\sigma(\pi) &\leq (i-1)(n-1) + (2r+1-i+1)(2r-i-1) \\
&\quad + (r-2)(n+1-2r-2) \\
&= i^2 + i(n-4r-2) - (n-1) \\
&\quad + (2r-1)(2r+2) + (r-2)(n-2r-1).
\end{aligned}$$

Since  $n \geq 4r + 10$ , it is easy to see that  $i^2 + i(n-4r-2)$ , consider as a function of  $i$ , attains its maximum value when  $i = r - 2$ . Therefore,

$$\begin{aligned}
\sigma(\pi) &\leq (r-2)^2 + (n-4r-2)(r-2) - (n-1) \\
&\quad + (2r-1)(2r+2) + (r-2)(n-2r-1) \\
&= (r-1)(2n-r) - 2(n-r) - n + 4r + 9 \\
&< \sigma(\pi),
\end{aligned}$$

which is a contradiction.

Thus,  $\sigma(K_{r+1} - T_3, n) \leq (r-1)(2n-r) - 2(n-r)$  for  $n \geq 4r+10$ .

**The Proof of Theorem 1.3** By Lemma 3.4, for  $r \geq 3, r+1 \geq k \geq 4$  and  $n \geq r+1$ ,  $\sigma(K_{r+1} - H, n) \geq (r-1)(2n-r) - 2(n-r)$ . Obviously, for  $r \geq 3, r+1 \geq k \geq 4$  and  $n \geq 4r+10$ ,  $\sigma(K_{r+1} - H, n) \leq \sigma(K_{r+1} - T_3, n)$ . By theorem 1.2, for  $r \geq 3, r+1 \geq k \geq 4$  and  $n \geq 4r+10$ ,  $\sigma(K_{r+1} - T_3, n) = (r-1)(2n-r) - 2(n-r)$ . Then  $\sigma(K_{r+1} - H, n) = (r-1)(2n-r) - 2(n-r)$ , for  $r \geq 3, r+1 \geq k \geq 4$  and  $n \geq 4r+10$ .

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